HARMONIC MAPPING PROBLEM AND AFFINE CAPACITY

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Abstract. The Harmonic Mapping Problem asks when there exists a harmonic homeomorphism between two given domains. It arises in the theory of minimal surfaces and in calculus of variations, specifically in hyperelasticity theory. We investigate this problem for doubly connected domains in the plane, where it already presents considerable challenge and leads to several interesting open questions.

1. Introduction

By virtue of Riemann Mapping Theorem for every pair (Ω, Ω^*) of simply connected domains in the complex plane one can find a conformal mapping $h\colon \Omega \xrightarrow{\text{onto}} \Omega^*$ except for two cases: $\Omega \subsetneq \mathbb{C} = \Omega^*$ or $\Omega^* \subsetneq \mathbb{C} = \Omega$. The situation is quite different for doubly connected domains.

A domain $\Omega \subset \mathbb{C}$ is doubly connected if $\widehat{\mathbb{C}} \setminus \Omega$ consists of two connected components; that is, continua in the Riemann sphere $\widehat{\mathbb{C}}$. We say that Ω is nondegenerate if both components contain more than one point. Every nondegenerate doubly connected domain can be conformally mapped onto an annulus

$$A(r, R) = \{z \colon r < |z| < R\}, \qquad 0 < r < R < \infty$$

where the ratio R/r does not depend on the choice of the conformal mapping. This gives rise to the notion of conformal modulus,

$$\operatorname{Mod} \Omega = \log \frac{R}{r}. \tag{1.1}$$

In fact, modulus is the only conformal invariant for nondegenerate doubly connected domains. Let us set $\operatorname{Mod} \Omega = \infty$ for the degenerate cases.

Complex harmonic functions, whose real and imaginary parts need not be coupled by the Cauchy-Riemann system, provide significantly larger class of mappings, but still restrictions on the domains are necessary. The studies

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of the Harmonic Mapping Problem began with Radó's theorem [21] which states that there is no harmonic homeomorphism $h\colon \Omega \xrightarrow{\text{onto}} \mathbb{C}$ for any proper domain $\Omega \subsetneq \mathbb{C}$. The first proof of Radó's theorem in this form was given by Bers [4]; both Radó and Bers were motivated by the celebrated Bernstein's theorem: *any global solution* (in the entire plane) *of the minimal surface equation is affine.*

Harmonic Mapping Problem for doubly connected domains originated from the work of J.C.C. Nitsche on minimal surfaces. In 1962 he formulated a conjecture [19] which was recently proved by the present authors [16].

Theorem A. *A harmonic homeomorphism $h\colon A(r,R) \to A(r^*,R^*)$ between circular annuli exists if and only if*

$$\frac{R^*}{r^*} \geqslant \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right). \tag{1.2}$$

It is a simple matter to see that harmonic functions remain harmonic upon conformal change of the independent variable $z \in \Omega$. Therefore, Theorem A remains valid when the annulus $A(r,R)$ is replaced by any doubly connected domain $\Omega \subset \mathbb{C}$ of the same modulus $\log \frac{R}{r}$. The Nitsche bound (1.2) then reads as

$$\frac{R^*}{r^*} \geqslant \frac{1}{2}\left(e^{\operatorname{Mod}\Omega} + e^{-\operatorname{Mod}\Omega}\right) = \cosh \operatorname{Mod}\Omega. \tag{1.3}$$

The harmonicity of a mapping $h\colon \Omega \to \Omega^*$ is also preserved under affine transformations of the target. Thus it is natural to investigate necessary and sufficient conditions for the existence of h in terms of the conformal modulus of Ω and an affine invariant of the target Ω^*. This leads us to the concept of affine modulus.

A $\mathbb{C}$-affine automorphism of $\mathbb{C}$ is a mapping of the form $z \mapsto az + c$ with $a, c \in \mathbb{C}$, $a \neq 0$. An $\mathbb{R}$-affine automorphism of $\mathbb{C}$, or simply affine transformation, takes the form $z \mapsto az + b\bar{z} + c$ with determinant $|a|^2 - |b|^2 \neq 0$.

Definition 1.1. The *affine modulus* of a doubly connected domain $\Omega \subset \mathbb{C}$ is defined by

$$\operatorname{Mod}_@ \Omega = \sup\{\operatorname{Mod}\phi(\Omega); \quad \phi\colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C} \text{ affine}\}. \tag{1.4}$$

For an equivalent formulation and properties of the affine modulus see Section 2.

We can now state our main results: a necessary condition (Theorem 1.2) and sufficient condition (Theorem 1.4) for the existence of a harmonic homeomorphism $h\colon \Omega \xrightarrow{\text{onto}} \Omega^*$.

Theorem 1.2. *If $h\colon \Omega \to \Omega^*$ is a harmonic homeomorphism between doubly connected domains, and Ω is nondegenerate, then*

$$\frac{\operatorname{Mod}_@ \Omega^*}{\operatorname{Mod}\Omega} \geqslant \Phi(\operatorname{Mod}\Omega) \tag{1.5}$$

where $\Phi\colon (0,\infty) \to (0,1)$ *is an increasing function such that* $\Phi(\tau) \to 1$ *as* $\tau \to \infty$. *More specifically,*

$$\Phi(\tau) = \lambda\left(\coth\frac{\pi^2}{2\tau}\right), \quad \text{where } \lambda(t) \geqslant \frac{\log t - \log(1+\log t)}{2+\log t},\ t \geqslant 1. \tag{1.6}$$

Conjecture 1.3. Based upon the Nitsche bound (1.3) it seems reasonable to expect that

$$\Phi(\tau) = \frac{1}{\tau}\log\cosh\tau, \qquad 0<\tau<\infty$$

but we have no proof of this.

Theorem 1.4. *Let* Ω *and* Ω^* *be doubly connected domains in* $\mathbb{C}$ *such that*

$$\operatorname{Mod}_{@}\Omega^* > \operatorname{Mod}\Omega. \tag{1.7}$$

Then there exists a harmonic homeomorphism $h\colon \Omega \to \Omega^*$ *unless* $\mathbb{C}\setminus\Omega^*$ *is bounded. In the latter case there is no such* h.

When $\operatorname{Mod}\Omega \to \infty$, the comparison of inequalities (1.5) and (1.7) shows that both are asymptotically sharp. We do not know if equality in (1.7) (with both sides finite) would suffice as well. This is discussed in Remark 2.4.

We write $\Omega_1 \overset{\sim}{\hookrightarrow} \Omega_2$ when Ω_1 is contained in Ω_2 in such a way that the inclusion map $\Omega_1 \hookrightarrow \Omega_2$ is a homotopy equivalence. For doubly connected domains this simply means that $\Omega_2^- \subset \Omega_1^-$ and $\Omega_1^+ \subset \Omega_2^+$. The monotonicity of modulus can be expressed by saying that $\Omega_1 \overset{\sim}{\hookrightarrow} \Omega_2$ implies $\operatorname{Mod}\Omega_1 \leqslant \operatorname{Mod}\Omega_2$ and $\operatorname{Mod}_{@}\Omega_1 \leqslant \operatorname{Mod}_{@}\Omega_2$.

Observe that both conditions (1.5) and (1.7) are preserved if Ω is replaced by a domain with a smaller conformal modulus, or Ω^* is replaced by a domain with a greater affine modulus. Thus, Theorems 1.2 and 1.4 suggest the formulation of the following comparison principles.

Problem 1.5. (Domain Comparison Principle) Let Ω and Ω^* be doubly connected domains such that Ω is nondegenerate and there exists a harmonic homeomorphism $h\colon \Omega \xrightarrow{\text{onto}} \Omega^*$. If $\Omega_\circ \overset{\sim}{\hookrightarrow} \Omega$, then there exists a harmonic homeomorphism $h_\circ\colon \Omega_\circ \xrightarrow{\text{onto}} \Omega^*$.

Problem 1.6. (Target Comparison Principle) Let Ω and Ω^* be doubly connected domains such that there exists a harmonic homeomorphism $h\colon \Omega \xrightarrow{\text{onto}} \Omega^*$. If $\Omega^*_\circ$ is nondegenerate and $\Omega^* \overset{\sim}{\hookrightarrow} \Omega^*_\circ$, then there exists a harmonic homeomorphism $h_\circ\colon \Omega \xrightarrow{\text{onto}} \Omega^*_\circ$.

Both Problems 1.5 and 1.6 are open to the best of our knowledge. Although the Harmonic Mapping Problem has its roots in the theory of minimal surfaces, it also arises from the minimization of the Dirichlet energy

$$\mathcal{E}[f] = \iint_\Omega |Df|^2$$

among all homeomorphism $f\colon \Omega \xrightarrow{\text{onto}} \Omega^*$. The minimizers and stationary points of $\mathcal{E}$ serve as a model of admissible deformations of a hyperelastic

material with stored energy $\mathcal{E}$ [18]. The existence of a harmonic homeomorphism $h\colon \Omega \xrightarrow{\text{onto}} \Omega^*$ is necessary for the minimum of $\mathcal{E}[f]$ to be attained in the class of homeomorphisms between Ω and Ω^*. Indeed, such minimal maps must be harmonic.

The rest of the paper is organized as follows: we discuss relevant affine invariants in Section 2, prove Theorem 1.2 in Section 3, and prove Theorem 1.4 in Section 4. We write $\mathbb{D}_r = \{z \in \mathbb{C}\colon |z| < r\}$, $\mathbb{T}_r = \partial \mathbb{D}_r$, and $\mathbb{T} = \mathbb{T}_1$.

The Harmonic Mapping Problem for multiply connected domains has been also studied in [5, 6, 7, 9, 10, 15, 17], where the authors mostly focus on existence of harmonic mappings onto some domain of a given canonical type (such as disk with punctures) rather than onto a specific domain.

2. Affine capacity and modulus

In this section Ω is a doubly connected domain in $\mathbb{C}$, possibly degenerate. Exactly one of the components of $\mathbb{C} \setminus \Omega$ is bounded and is denoted Ω^-. We also write $\Omega^+ = \Omega^- \cup \Omega$, which is a simply connected domain. Let $|E|$ denote the area (planar Lebesgue measure) of a set $E \subset \mathbb{C}$. We are now ready to introduce the first affine invariant of Ω.

Definition 2.1. The *Carleman modulus* of Ω is defined by the rule

$$\operatorname{Mod}_{\copyright} \Omega = \frac{1}{2} \log \frac{|\Omega^+|}{|\Omega^-|} = \frac{1}{2} \log \left(1 + \frac{|\Omega|}{|\Omega^-|}\right)$$

unless $|\Omega| = \infty$ or $|\Omega^-| = 0$, in which case $\operatorname{Mod}_{\copyright} \Omega := \infty$.

The well-known inequality

$$\operatorname{Mod} \Omega \leqslant \operatorname{Mod}_{\copyright} \Omega, \tag{2.1}$$

proved by T. Carleman [8] in 1918, can be expressed by saying that among all doubly connected domains with given areas of Ω^- and Ω^+ the maximum of conformal modulus is uniquely attained by the circular annulus $A(r, R)$, $\pi r^2 = |\Omega^-| < |\Omega^+| = \pi R^2$. Carleman's inequality was one of the earliest isoperimetric-type results in mathematical physics, many of which can be found in [20].

Our second affine invariant arises from an energy minimization problem. Recall the capacity of Ω,

$$\operatorname{Cap} \Omega = \inf_u \iint_{\mathbb{C}} |\nabla u|^2 \tag{2.2}$$

where the infimum is taken over all real-valued smooth functions on $\mathbb{C}$ which assume precisely two values 0 and 1 on $\mathbb{C} \setminus \Omega$. The conformal modulus, defined in (1.1), is given by

$$\operatorname{Mod} \Omega = \frac{2\pi}{\operatorname{Cap} \Omega}.$$

Since we are looking for an affine invariant of Ω, the following variational problem is naturally introduced.

Definition 2.2. Define the *affine capacity* of Ω by

$$\operatorname{Cap}_{@} \Omega := \inf_{A,u} \frac{1}{|\det A|} \iint_{\Omega} |A \nabla u|^2$$

where the infimum is taken over all invertible matrices A and over all real functions u as in (2.2). Thus, the affine modulus of Ω, defined in (1.4), is

$$\operatorname{Mod}_{@} \Omega = \frac{2\pi}{\operatorname{Cap}_{@} \Omega}.$$

Let us now examine the properties of the affine modulus. From (2.1) we see that

$$\operatorname{Mod} \Omega \leqslant \operatorname{Mod}_{@} \Omega \leqslant \operatorname{Mod}_{©} \Omega. \tag{2.3}$$

When Ω is a circular annulus, equality holds in (2.1) and therefore in (2.3). Hence $\operatorname{Mod}_{@} A(r,R) = \log(R/r)$.

Equality $\operatorname{Mod}_{@} \Omega = \operatorname{Mod} \Omega$ is also attained, for example, if Ω is the *Teichmüller ring*

$$\mathcal{T}(s) := \mathbb{C} \setminus ([-1,0] \cup [s,+\infty)), \qquad s > 0$$

Indeed, for any affine automorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ there is a $\mathbb{C}$-affine automorphism $\psi\colon \mathbb{C} \to \mathbb{C}$ that agrees with ϕ on $\mathbb{R}$. Since $\phi(\mathcal{T}(s)) = \psi(\mathcal{T}(s))$ and ψ is conformal, it follows that

$$\operatorname{Mod} \phi(\mathcal{T}(s)) = \operatorname{Mod} \mathcal{T}(s). \tag{2.4}$$

On the other extreme, $\operatorname{Mod}_{@} \Omega$ may be infinite even when $\operatorname{Mod} \Omega$ is finite. Indeed, the domain

$$\Omega = \{z \in \mathbb{C} \colon |\operatorname{Im} z| < 1\} \setminus [-1,1]$$

is affinely equivalent to $\{z \in \mathbb{C} \colon |\operatorname{Im} z| < 1\} \setminus [-s,s]$ for any $s > 0$. The conformal modulus of the latter domain grows indefinitely as $s \to 0$.

Since the second inequality in (2.3) becomes vacuous when $|\Omega| = \infty$ or $|\Omega^-| = 0$, it is desirable to have an upper estimate for $\operatorname{Mod}_{@} \Omega$ in terms of other geometric properties of Ω. Recall that the *width* of a compact set $E \subset \mathbb{C}$, denoted $w(E)$, is the smallest distance between two parallel lines that enclose the set. For connected sets this is also the length of the shortest 1-dimensional projection.

Lemma 2.3. *Let Ω be a nondegenerate doubly connected domain. If $w = w(\Omega^-) > 0$, then*

$$\operatorname{Mod}_{@} \Omega \leqslant \operatorname{Mod} \mathcal{T}(d/w), \qquad \text{where } d = \operatorname{dist}(\partial\Omega^+, \Omega^-). \tag{2.5}$$

Proof. Let $\phi\colon \mathbb{C} \to \mathbb{C}$ be an affine automorphism. Denote its Lipschitz constant by $L := |\phi_z| + |\phi_{\bar z}|$. Clearly $\operatorname{dist}(\partial\phi(\Omega^+), \phi(\Omega^-)) \leqslant L\, d$ and $\operatorname{diam} \phi(\Omega^-) \geqslant Lw$. Now (2.5) follows from the extremal property of the Teichmüller ring: it has the greatest conformal modulus among all domains

with given diameter of the bounded component and given distance between components [1, Ch. III.A]. □

Even when the affine modulus is finite, the supremum in (1.4) is not always attained. An example is given by the *Grötzsch ring*

$$\mathcal{G}(s) = \{z \in \mathbb{C} \colon |z| > 1\} \setminus [s, +\infty), \qquad s > 1.$$

Indeed,

$$\operatorname{Mod}_{@} \mathcal{G}(s) \leqslant \operatorname{Mod} \mathcal{T}\left(\tfrac{s-1}{2}\right) \tag{2.6}$$

by Lemma 2.3. Equality holds in (2.6) because the images of $\mathcal{G}(s)$ under mappings of the form $z + k\bar{z}$, $k \nearrow 1$, converge to the domain $\mathbb{C} \setminus ([-2, 2] \cup [2s, \infty))$ which is a $\mathbb{C}$-affine image of $\mathcal{T}\left(\frac{s-1}{2}\right)$. Yet, for any affine automorphism ϕ

$$\operatorname{Mod} \phi(\mathcal{G}(s)) < \operatorname{Mod} \phi\bigl\{\mathbb{C} \setminus ([-1, 1] \cup [s, \infty))\bigr\} = \operatorname{Mod} \mathcal{T}\left(\tfrac{s-1}{2}\right)$$

where the first part expresses the monotonicity of modulus, and the second follows from (2.4). Thus the supremum in (1.4) is not attained by any ϕ.

Remark 2.4. Since equality holds in (2.6), Lemma 2.3 is sharp. Furthermore, the pair of domains $\Omega = \mathcal{T}\left(\frac{s-1}{2}\right)$ and $\Omega^* = \mathcal{G}(s)$ can serve as a test case for whether equality in (1.7) implies the existence of a harmonic homeomorphism.

Remark 2.5. The identity

$$\operatorname{Mod}_{@} \mathcal{G}(s) = \operatorname{Mod}_{@} \mathcal{T}\left(\tfrac{s-1}{2}\right) \tag{2.7}$$

somewhat resembles the relation between conformal moduli of the Grötzsch and Teichmüller rings [1, Ch. III.A],

$$\operatorname{Mod} \mathcal{G}(s) = \frac{1}{2} \operatorname{Mod} \mathcal{T}(s^2 - 1).$$

3. Proof of Theorem 1.2

Before proceeding to the proof we recollect basic facts of potential theory in the plane which can be found in [22]. A domain Ω has Green's function G_Ω whenever $\mathbb{C} \setminus \Omega$ contains a nondegenerate continuum. Our normalization is $G_\Omega(z, \zeta) = -\log|z - \zeta| + O(1)$ as $z \to \zeta$. In particular, $G_\Omega(z, \zeta) > 0$. Green's function for the unit disk is

$$G_{\mathbb{D}}(z, \zeta) = \log \left| \frac{1 - z\bar{\zeta}}{z - \zeta} \right|. \tag{3.1}$$

If $f \colon \Omega \to \Omega^*$ is a holomorphic function, then the subordination principle holds:

$$G_\Omega(z, \zeta) \leqslant G_{\Omega^*}(f(z), f(\zeta)). \tag{3.2}$$

One consequence of (3.1) and (3.2) is a general version of the Schwarz lemma. If $f \colon \Omega \to \mathbb{D}$ is holomorphic and $f(a) = 0$ for some $a \in \Omega$, then

$$|f(z)| \leqslant \exp(-G_\Omega(z, a)) < 1, \qquad z \in \Omega. \tag{3.3}$$

There is an application of (3.3) to harmonic homeomorphisms. We refer to [9, p.5] for a discussion of relation between harmonic and quasiconformal mappings, and to the book [1] for the general theory of quasiconformal mappings.

Proposition 3.1. *Let $\Omega \subset \mathbb{C}$ be a domain with Green's function G_Ω. Fix $a \in \Omega$. For any harmonic homeomorphism $h\colon \Omega \to \mathbb{C}$ there exists an affine automorphism ϕ such that the composition $H = \phi \circ h$ satisfies*

$$\frac{|H_{\bar z}|}{|H_z|} \leqslant \exp(-G_\Omega(z,a)) < 1 \tag{3.4}$$

for $z \in \Omega$.

Proof. Replacing h with $\bar h$, we may assume that h is orientation preserving. Then h satisfies the second Beltrami equation

$$h_{\bar z}(z) = \nu(z)\overline{h_z(z)} \tag{3.5}$$

where the *second complex dilatation* ν is an antiholomorphic function from Ω into ∂. Since $|\nu(z)| < 1$, the mapping h is quasiconformal away from the boundary of Ω. Affine transformations of h do not affect harmonicity by may decrease $|\nu|$. Indeed, the composition

$$H(z) = h(z) - \varkappa \bar h(z), \qquad \varkappa \in \mathbb{D},\ \alpha \in \mathbb{C} \setminus \{0\}$$

satisfies new second Beltrami equation

$$H_{\bar z}(z) = \frac{\nu(z) - \varkappa}{1 - \overline{\varkappa}\nu(z)}\overline{H_z(z)} \tag{3.6}$$

By setting $\varkappa = \nu(a)$ we achieve that the second complex dilatation of H vanishes at a. Now the required estimate (3.4) follows from (3.3). □

We would like to make the estimate (3.4) more explicit when the domain Ω is a circular annulus $\Omega = A(R^{-1}, R)$. Although an explicit formula for Green's function of an annulus can be found, for us it suffices to have a lower bound. We obtain such a bound from the subordination principle (3.2). Indeed, for any $\alpha > 0$ the vertical strip $S = \{z \in \mathbb{C}\colon |\operatorname{Re} z| < \frac{\pi}{2\alpha}\}$ has Green's function [22, p. 109]

$$G_S(z,\zeta) = \log\left|\frac{e^{i\alpha z} + e^{-i\alpha\bar\zeta}}{e^{i\alpha z} - e^{i\alpha\zeta}}\right|. \tag{3.7}$$

We use (3.7) with $\zeta = 0$ and $z = iy$ where $|y| \leqslant \pi$

$$G_S(iy,0) = \log\left|\frac{e^{-\alpha y} + 1}{e^{-\alpha y} - 1}\right| = \log\coth\frac{\alpha|y|}{2} \geqslant \log\coth\frac{\pi\alpha}{2}. \tag{3.8}$$

Since $w = e^z$ maps S onto $\Omega = A(R^{-1}, R)$ with $\operatorname{Mod}\Omega = 2\log R = \pi/\alpha$, inequality (3.8) together with the subordination principle (3.2) yield

$$G_\Omega(z,\zeta) \geqslant \log\coth\frac{\pi^2}{4\log R}, \qquad z,\zeta \in \mathbb{T}. \tag{3.9}$$

We are now ready to give an explicit estimate for holomorphic functions with at least one zero in an annulus.

Lemma 3.2. *Let $\Omega = A(R^{-1}, R)$ and suppose that $f\colon \Omega \to \mathbb{D}$ is a holomorphic function with $f(1) = 0$. Then for each $0 \leqslant \alpha < 1$ we have*

$$\max_{R^{-\alpha}\leqslant |z| \leqslant R^{\alpha}} |f(z)| \leqslant k^{1-\alpha}, \qquad k = \tanh \frac{\pi^2}{4\log R} < 1 \tag{3.10}$$

Proof. The case $\alpha = 0$ immediately follows from (3.3) and (3.9). For the general case, let

$$M(r) = \max_{|z|=r} |f(z)|, \qquad \frac{1}{R} < r < R$$

By Hadamard's three circle theorem, $\log M(r)$ is a convex function of $\log r$. Since $\log M(1) \leqslant \log k$ and $\log M(r) < 1$ for all r, the convexity implies

$$\log M(R^{\alpha}) \leqslant (1-\alpha)\log k$$

and the same estimate holds for $\log M(R^{-\alpha})$. □

Proof of Theorem 1.2. With the aid of conformal transformation of Ω we may assume that $\Omega = A(R^{-1}, R)$ where $2\log R = \operatorname{Mod}\Omega$. As in the proof of Proposition 3.1, we apply an affine transformation to obtain a harmonic mapping H whose second complex dilatation vanishes at 1. By Lemma 3.2 the restriction of H to $\Omega_\alpha = A(R^{-\alpha}, R^{\alpha})$ is K-quasiconformal; that is,

$$\frac{|H_z| + |H_{\bar z}|}{|H_z| - |H_{\bar z}|} \leqslant K = \frac{1+k^{1-\alpha}}{1-k^{1-\alpha}}, \qquad k = \tanh \frac{\pi^2}{4\log R}.$$

The conformal modulus of Ω_α is distorted by the factor of at most K under the mapping H, see [1]. Hence

$$\operatorname{Mod}_{@} h(\Omega) \geqslant \operatorname{Mod} H(\Omega) \geqslant \frac{1-k^{1-\alpha}}{1+k^{1-\alpha}} \operatorname{Mod}\Omega_\alpha = \alpha\,\frac{1-k^{1-\alpha}}{1+k^{1-\alpha}} \operatorname{Mod}\Omega. \tag{3.11}$$

We are free to choose any $0 < \alpha < 1$ in (3.11). Introduce the function

$$\lambda(t) = \sup_{0<\alpha<1} \alpha\,\frac{t^{1-\alpha}-1}{t^{1-\alpha}+1}, \qquad t \geqslant 1 \tag{3.12}$$

which is positive and decreasing for $t > 0$. Clearly $\lambda(t) \to 1$ as $t \to \infty$. Now (3.11) takes the form

$$\frac{\operatorname{Mod}_{@} h(\Omega)}{\operatorname{Mod}\Omega} \geqslant \Phi(\operatorname{Mod}\Omega), \qquad \text{where } \Phi(\tau) = \lambda\Big(\coth \frac{\pi^2}{2\tau}\Big) \tag{3.13}$$

and $\Phi(\tau) \to 1$ as $\tau \to \infty$. To obtain a concrete bound, we test the supremum in (3.12) by putting

$$\alpha = 1 - \frac{\log(1+\log t)}{\log t},$$

obtaining

$$\lambda(t) \geqslant \left(1 - \frac{\log(1+t)}{t}\right) \frac{\log t}{2 + \log t}$$

which is (1.6). □

4. Proof of Theorem 1.4

For notational simplicity we denote by $\mathcal{H}(\Omega, \Omega^*)$ the class of harmonic homeomorphisms from Ω onto Ω^*. This includes conformal mappings $\Omega \to \Omega^*$ if they exist. The proof of Theorem 1.4 is divided into three parts.

4.1. **Exceptional Pairs.** In this section we prove that $\mathcal{H}(\Omega, \Omega^*)$ is empty when $\operatorname{Mod} \Omega < \infty$ and $\mathbb{C} \setminus \Omega^*$ is bounded. Suppose to the contrary that $h \in \mathcal{H}(\Omega, \Omega^*)$. Up to a conformal transformation, Ω is the circular annulus $A(1, R)$. The target domain is $\Omega^* = \mathbb{C} \setminus E$ for some compact set E. With the help of inversion $z \mapsto R^2/z$ we can arrange so that $h(z)$ approaches E as $|z| \to 1$.

For large enough integers m the open disk $\mathbb{D}_m$ contains E. Let $\Omega_m = h^{-1}(\mathbb{D}_m \setminus E)$. As $m \to \infty$, the outer boundary of Ω_m approaches $\mathbb{T}_R$. Thus there exist conformal mappings $g_m \colon A(1, R_m) \xrightarrow{\text{onto}} \Omega_m$ such that $g_m(z) \to z$ pointwise and $R_m \nearrow R$ as $m \to \infty$. Define a harmonic mapping

$$h_m(z) = \frac{1}{m} h(g_m(R_m z)), \qquad z \in A(R_m^{-1}, 1)$$

and observe that for any $z \in A(R^{-1}, 1)$ the value $h_m(z)$ is defined when m is large enough. Moreover, $h_m(z) \to 0$ because $g_m(R_m z) \to Rz$, and the convergence is uniform on compact subsets of $A(R^{-1}, 1)$, i.e.,

$$\lim_{m \to \infty} \sup_{r_1 \leqslant |z| \leqslant r_2} |h_m(z)| \to 0, \qquad R^{-1} < r_1 < r_2 < 1. \tag{4.1}$$

Let us write

$$\begin{aligned} h_m(z) &= \sum_{n \neq 0} \left(a_n z^n + b_n \bar{z}^{-n}\right) + a_0 \log|z| + b_0 \\ &= \sum_{n \neq 0} \left(a_n \rho^n + b_n \rho^{-n}\right) e^{ni\theta} + a_0 \log \rho + b_0, \quad z = \rho e^{i\theta} \end{aligned} \tag{4.2}$$

where the coefficients depend on m as well but we suppress this dependence in the notation. In fact, (4.1) implies that for each fixed n the coefficients a_n, b_n tend to 0 as $m \to \infty$. On the other hand, h_m extends to a sense-preserving homeomorphism $\mathbb{T} \to \mathbb{T}$, which leads to a contradiction with the following result. □

Theorem 4.1. *(Weitsman [25]) Let $f \colon \mathbb{T} \to \mathbb{T}$ be a sense preserving homeomorphism with Fourier coefficients c_n, $n \in \mathbb{Z}$. Then*

$$|c_0| + |c_1| \geqslant \frac{2}{\pi}. \tag{4.3}$$

Weitsman's inequality is sharp. Earlier Hall [11] proved (4.3) with 1/2 instead of $2/\pi$. The validity of (4.3) with some absolute constant can be traced to the work of several authors, see [12]. It is worth mentioning that (4.3) first arose as a special case of Shapiro's conjecture [2, Problem 5.41], which posed that for any sense-preserving k-fold cover $f\colon \mathbb{T} \to \mathbb{T}$

$$|c_0|^2 + |c_1|^2 + \cdots + |c_k|^2 \geqslant \delta_k \tag{4.4}$$

where $\delta_k > 0$ depends only on k. This conjecture was proved by Hall [11, 12] for $k = 2$ and by Sheil-Small [23] in general. The rate of decay of δ_k remains unknown, see [13].

4.2. **Non-exceptional pairs with degenerate target.** Here we assume that $\operatorname{Mod}\Omega^* = \infty$ but $\mathbb{C} \setminus \Omega^*$ is unbounded. Without loss of generality, $\Omega^* = G \setminus \{0\}$ where $G \subsetneq \mathbb{C}$ is a simply connected domain and $0 \in G$. For $t \geqslant 0$ we define a mapping $F_t\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by

$$F_t(re^{i\theta}) = (r + \sqrt{r^2 + t^2})e^{i\theta}$$

Note that $|F_t(z)| \geqslant t$ for all $z \neq 0$. The choice of F_t is motivated by the fact that its inverse is harmonic:

$$F_t^{-1}(\zeta) = \frac{1}{2}\left(\zeta - \frac{t^2}{\bar{\zeta}}\right)$$

Since F_t^{-1} maps the doubly connected domain $F_t(\Omega^*)$ onto Ω^*, it remains to find t such that $\operatorname{Mod} F_t(\Omega^*) = \operatorname{Mod}\Omega$. The latter will follow from the intermediate value theorem once we prove $\operatorname{Mod} F_t(\Omega^*) \to 0$ as $t \to \infty$ and $\operatorname{Mod} F_t(\Omega^*) \to \infty$ as $t \to 0$. The latter is obvious, so we proceed to the proof of the former limit.

Let $d = \operatorname{dist}(\partial G, 0)$. The complement of $F_t(\Omega^*)$ has two components: one is the disk $\overline{\mathbb{D}_t}$ and the other contains a point with absolute value $d + \sqrt{d^2 + t^2}$. By the extremal property of the Grötzsch ring [1, Ch. III.A], $\operatorname{Mod} F_t(\Omega^*)$ does not exceed the conformal modulus of the Grötzsch ring $\mathcal{G}(s)$ with

$$s = \frac{d + \sqrt{d^2 + t^2}}{t}$$

As $t \to \infty$, we have $s \to 1$ and thus $\operatorname{Mod}\mathcal{G}(s) \to 0$. This completes the proof. □

4.3. **Non-exceptional pairs with nondegenerate target.** Since harmonicity is invariant under affine transformations of the target, we may assume that $\operatorname{Mod}\Omega^* > \operatorname{Mod}\Omega$. There are two substantially different cases.

Case 1. The set $\mathbb{C} \setminus \Omega^*$ is contained in a line. Thus, up to a $\mathbb{C}$-affine automorphism, Ω^* is the Teichmüller ring

$$\mathcal{T}(t) = \mathbb{C} \setminus ([-1, 0] \cup [t, +\infty))$$

We may and do replace Ω with a conformally equivalent domain $\mathcal{T}(s)$ for some $0 < s < t$. Thus our task is to harmonically map $\mathcal{T}(s)$ onto $\mathcal{T}(t)$.

Let $b \geqslant 0$ be a number to be chosen later. Define a piecewise linear function $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} b, & x \leqslant -1 \\ -bx, & -1 \leqslant x \leqslant 0 \\ 0, & x \geqslant 0 \end{cases}$$

and consider the domain $G_b = \{x + iy\colon y > g(x)\}$ shown in Figure 1.

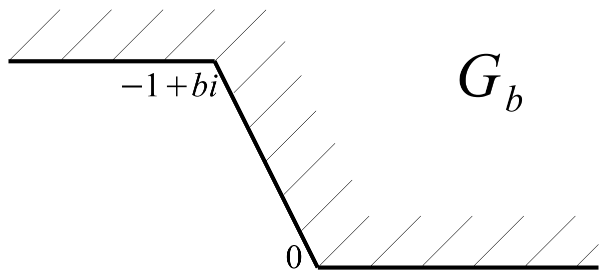

FIGURE 1. Polygonal domain G_b

Let $f = f_b$ be a conformal mapping of the upper halfplane $\mathbb{H} = \{z\colon \operatorname{Im} z > 0\}$ onto G_b normalized by boundary conditions $f(-1) = -1 + bi$, $f(0) = 0$, and $f(\infty) = \infty$. It is important to notice that the boundary of G_b satisfies the quasiarc condition uniformly with respect to b; that is,

$$\left| \frac{\zeta_2 - \zeta_1}{\zeta_2 - \zeta_1} \right| \leqslant C \tag{4.5}$$

for any three points on ∂G_b such that ζ_3 separates ζ_1 and ζ_2. By a theorem of Ahlfors [1, p. 49] f extends to a K-quasiconformal mapping $\mathbb{C} \to \mathbb{C}$ with K independent of b. The latter can be expressed via the quasisymmetry condition (see [24] or [14, Ch. 11]): there is a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$, independent of b, such that

$$\frac{|f(p) - f(q)|}{|f(p) - f(r)|} \leqslant \eta \left(\frac{|p - q|}{|p - r|} \right) \tag{4.6}$$

for all distinct points $p, q, r \in \mathbb{C}$. Applying (4.6) to the triple $-1, 0, s$, we find

$$f(s) \geqslant \frac{\sqrt{1 + b^2}}{\eta(1/s)} \tag{4.7}$$

Hence $f(s) \to \infty$ as $b \to \infty$. On the other hand, $f(s) = s$ when $b = 0$ Since $f(s) = f_b(s)$ depends continuously on b, there exists $b > 0$ for which $f(s) = t$. Let us fix such b.

As observed above, f_b has a continuous extension to $\overline{\mathbb{H}}$. It takes the segments $(-\infty, -1)$ and $(0, s)$ into horizontal segments $(-\infty, -1) + ib$ and

$(0,t)$ respectively. By the reflection principle f_b extends holomorphically across each segment, and we have $\operatorname{Re} f(\bar z) = \operatorname{Re} f(z)$. It follows that the function $u(z) = \operatorname{Re} f(z)$ extends harmonically to the entire Teichmüller ring $\mathcal{T}(s)$.

Consider the harmonic mapping

$$h(z) = u(z) + i \operatorname{Im} z \tag{4.8}$$

which by construction is continuous in $\mathbb{C}$. We claim that h is a homeomorphism from $\mathcal{T}(s)$ onto $\mathcal{T}(t)$. Since h agrees with f on $\mathbb{R}$, it follows that h maps $\mathbb{R}$ homeomorphically onto $\mathbb{R}$ in such a way that $h(-1) = -1$, $h(0) = 0$, and $h(s) = t$. To prove that h is injective in $\mathcal{T}(s)$, we only need to show that $u(x,y)$ is a strictly increasing function of x for any fixed $y > 0$. The partial derivative u_x is harmonic and nonconstant in $\mathbb{H}$, has nonnegative boundary values on $\mathbb{R}$, and is bounded at infinity. Thus $u_x > 0$ in $\mathbb{H}$ by the maximum principle [22] and the claim is proved.

Case 2. The set $\mathbb{C} \setminus \Omega^*$ is not contained in a line. We claim that there exists an affine automorphism ϕ such that $\operatorname{Mod} \phi(\Omega^*) = \operatorname{Mod} \Omega$. If this holds, then Ω can be mapped conformally onto $\phi(\Omega^*)$ and the composition with ϕ^{-1} furnishes the desired harmonic homeomorphism. Since $\operatorname{Mod} \Omega^* > \operatorname{Mod} \Omega$ and $\operatorname{Mod} \phi(\Omega^*)$ depends continuously on the coefficients of ϕ, it suffices to show

$$\inf_{\phi} \{\operatorname{Mod} \phi(\Omega^*); \quad \phi \colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C} \text{ affine}\} = 0 \tag{4.9}$$

Let E and F be the bounded and unbounded components of the complement of Ω^*, respectively. We need a lemma.

Lemma 4.2. *There exists a compact set $\tilde F \subset F$ such that the union $E \cup \tilde F$ is not contained in a line.*

Proof. Let $F_R = \{z \in F \colon |z| \leqslant R\}$ for $R > 0$. If E is not contained in any line, then we can let $\tilde F$ be any nontrivial connected component of F_R, with R large. If E is contained in some line ℓ, then F_R is not contained in ℓ when R is large enough, and we can again choose $\tilde F$ to be one of its connected components. □

There exist two parallel lines ℓ_1 and ℓ_2 such that any line between ℓ_1 and ℓ_2 meets both E and $\tilde F$. We may assume that these lines are vertical, say $\operatorname{Re} z = 0$ and $\operatorname{Re} z = a$. Let $D = \operatorname{diam}(E \cup \tilde F)$. For each $0 < t < a$ there is a segment of line $\operatorname{Re} z = t$ that joins E to F and has length is at most D. The extremal length of the family of all such segments is at most D/a, see [1, Ch. I]. Under the affine transformation $\phi(x+iy) = Mx + iy$ the length of vertical segments is unchanged, while the width of their family becomes Ma instead of a. Thus the extremal length of all rectifiable curves connecting $\phi(E)$ and $\phi(\tilde F)$ tends to 0 as $M \to \infty$. The relation between modulus and extremal length [1, Ch. I] implies $\operatorname{Mod} \phi(\Omega^*) \to 0$ as $M \to \infty$. This completes the proof of (4.9) and of Theorem 1.4. □

5. Concluding remarks

Since Conjecture 1.3 is known to be true for circular annuli [16] (Nitsche Conjecture), it is natural to test it on other canonical doubly connected domains: the Grötzsch and Teichmüller rings. Precisely, the questions are as follows.

Question 5.1. *Grötzsch–Nitsche Problem*: for which $1 < s, t < \infty$ does there exist a harmonic homeomorphism $h\colon \mathcal{G}(s) \xrightarrow{\text{onto}} \mathcal{G}(t)$?
Teichmüller–Nitsche Problem: for which $0 < s, t < \infty$ does there exist a harmonic homeomorphism $h\colon \mathcal{T}(s) \xrightarrow{\text{onto}} \mathcal{T}(t)$?

We offer the following observation.

Remark 5.2. There exists a harmonic homeomorphism $h\colon \mathcal{T}(s) \xrightarrow{\text{onto}} \mathcal{T}(t)$ provided that

$$\frac{t+1}{t} \leqslant \Big(\frac{s+1}{s}\Big)^{3/2}. \tag{5.1}$$

Indeed, by virtue of Theorem 1.4 we only need to consider $t < s$. Choose $1 < \alpha \leqslant 3/2$ so that

$$\frac{t+1}{t} = \Big(\frac{s+1}{s}\Big)^{\alpha}.$$

Let $G = \mathbb{C} \setminus (-\infty, 0]$. Choose a branch of $f(z) = z^\alpha$ in G so that $f(1) = 1$. Note that $\operatorname{Re} f' > 0$ in G. Therefore, the harmonic function $u(z) = \operatorname{Re} f(z)$ satisfies $u_x > 0$ in G. It follows that $h(z) := u(z) + i \operatorname{Im} z$ is a harmonic homeomorphism of G onto itself. It remains to observe that h maps the domain $G \setminus [s, s+1]$ onto $G \setminus [s^\alpha, (s+1)^\alpha]$. □

Numerical computations with conformal moduli of Teichmüller rings [1, 3] show that Example 5.2 does not contradict Conjecture 1.3.

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